discovery

with Neville de Mestre

Triangle inequality revisited

If you missed the recent great article in the *Australian Mathematics Teacher* (2015) by Pat Graham and Helen Chick, I strongly recommend that you obtain a copy and put it into practice in your classroom. As outlined in their paper, an incredibly useful set of information about the mathematical ability of your students will be revealed. You can look at the way your students try to solve the 20 matchsticks problem, their method of recording, their errors, and also the ramifications of the questions that they ask. When your students have completed and discussed the 20 matchsticks problem, they should be ready to tackle the following extensions.

The first extension is to ask them to investigate the possible triangles (a, b, c) that have 1, 2, 3, …, N matchsticks of equal length as the longest side of the triangle rather than the total perimeter. It is still required that each of the other two sides is exactly an integer number of matchsticks.

For $N = 1$ there is only (1, 1, 1). For $N = 2$ there are (2, 2, 2) and (1, 2, 2), and to avoid any duplication all acceptable triangles (a, b, N) have $a <= b$, $<= N$. Note that (1, 1, 2) is not possible, since we are not including degenerate triangles and the triangle inequality $a + b > N$ is being strictly applied. With this start, your students should now explore further with extra matchsticks and produce :

$N =$ 3, (1, 3, 3) (2, 2, 3) (2, 3, 3) (3, 3, 3)
$N =$ 4, (1, 4, 4) (2, 3, 4) (2, 4, 4) (3, 3, 4)
(3, 4, 4) (4 ,4, 4)

To help with discovering the pattern, it now transpires that the results should be re-ordered so that each triangle has the form (N, b, a) with $N >= b >= a$ that is, its sides are given in descending order. Thus:

$N =$ 3, (3, 3, 3) (3, 3, 2) (3, 3, 1) (3, 2, 2)
$N =$ 4, (4, 4, 4) (4, 4, 3) (4, 4, 2) (4, 4, 1)
(4, 3, 3) (4, 3, 2)

Here an important point in finding patterns has emerged. For now it is seen that the pattern for the first four values of N is 1, 2, 3 + 1, 4 + 2, compared with 1, 2, 4, 6 for the total number of triangles. As soon as $N = 5$ and $N = 6$ are investigated the advantage of displaying the pattern this way is revealed. Your students should hopefully produce the following table for the total triangles for N up to 8.

N	1	2	3	4	5	6	7	8
Triangles	1	2	4	6	9	12	16	20

Further investigation reveals that for $N = 2k - 1$ (odd) the total number of triangles is $S(k) = k^2 = T(k-1) + T(k)$, while for $N = 2k$ (even) the total number of triangles is $2T(k) = k(k + 1)$. Here S denotes the square numbers and T denotes the triangular numbers.

The next extension is to find the number of possible triangles when the total number of matchsticks for the two shorter sides is N. That is, $a + b = N$ is given, and the acceptable values for c are required that once again satisfy the triangle inequality. Again, all sides have to be an integer number of matchsticks. The simplest triangle is (1, 1, 1) for $N = 2$. For $N = 3$ there is again only one triangle (1, 2, 2). The next two values of N produce:

$N =$ 4, (1, 3, 3) (2, 2, 2) (2, 2, 3)
$N =$ 5, (1, 4, 4) (2, 3, 3) (2, 3, 4)

Note that to avoid duplication the possible integer sides are given this time in ascending order. Further investigation by your students should yield the following table:

N	2	3	4	5	6	7	8	9
Triangles	1	1	3	3	6	6	10	10

Therefore the general result is $T(k)$ for $N = 2k$ (even) and $T(k)$ for $N = 2k + 1$ (odd). Note that the triangular numbers are incorporated once again. Finally, your students should return to the original problem of the 20 matchsticks challenge and investigate it for all values of N from 3 to 24. The first few values of N should not cause many problems, and the results are tabulated here:

N	a = 1	a = 2	a = 3	a = 4	a = 5	a = 6	a = 7	a = 8	total
3	1								1
4									0
5	1								1
6		1							1
7	1	1							2
8		1							1
9	1	1							3
10		1							2
11	1	1							4
12		1	1						3
13	1	1	1						5

It is difficult to see a pattern here, but a little perseverance to $N = 24$ reveals a lot more.

N	a = 1	a = 2	a = 3	a = 4	a =5	a = 6	a = 7	a = 8	total
14		1	2						4
15	1	1	1	1					7
16		1	2	1					5
17	1	1	1	2					8
18		1	2	2	1				7
19	1	1	1	2	1				10
20		1	2	2	2				8
21	1	1	1	2	2	1			12
22		1	2	2	3	1			10
23	1	1	1	2	2	2			14
24		1	2	2	3	2	1		12

Although the pattern of the totals seems quite irregular, the pattern for the subsets of triangles within each total is much clearer. But for this problem the triangular numbers are not appearing. The solution has recently been published (de Mestre, 2016).

Happy further discoveries!

References

de Mestre, N. (2016). Diophantine triples: Meandering towards a solution to the N-matchstick challenge, ***Gazette of the Australian Mathematical Society, 43***(1), 20–24.

Graham, P. & Chick, H. (2015). The 20 matchstick triangle challenge; an activity to foster reasoning and problem solving", ***Australian Mathematics Teacher 71***(1), 9–17.